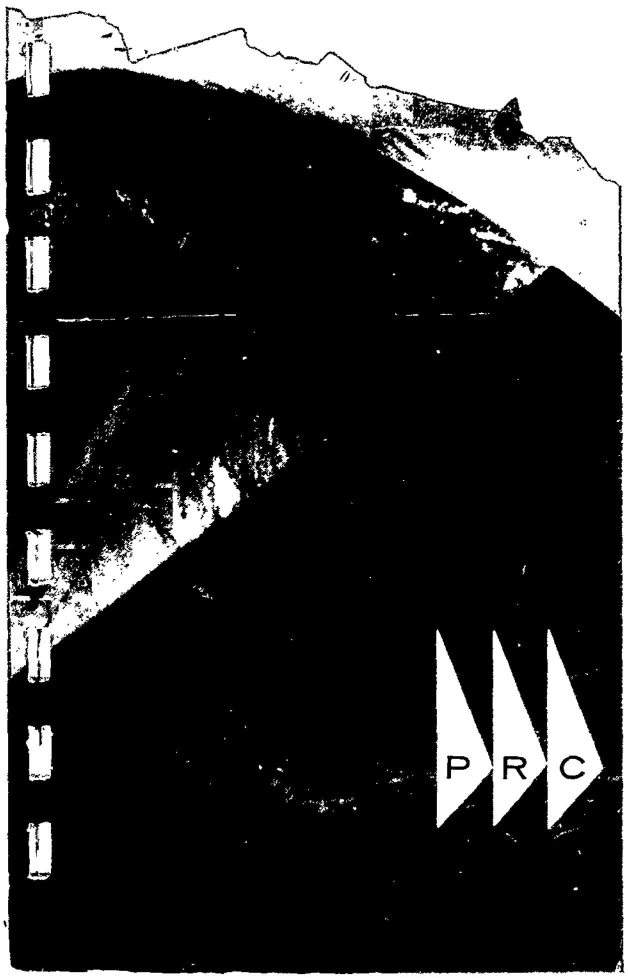
MATERIEL MANAGEMENT STUDIES

OPTIMAL POLICIES FOR THE INVENTORY PROBLEM WITH NEGOTIABLE LEADTIME



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OPTIMAL POLICIES FOR THE INVENTORY PROBLEM WITH NEGOTIABLE LEADTIME

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ABSTRACT

In this paper we study the dynamic inventory problem in which amounts of stock ordered at unit prices c_k and c_{k+1} $(c_k > c_{k+1})$ are delivered, respectively, k and $k+1$ periods later. It is demonstrated that under suitable cost conditions, the optimal policies are similar to those of the dynamic inventory problem with a delivery lag of $k+1$ periods, except for an additional constant stock level up to which it is desired to order at unit price c_k. If we assume that ordering decisions are made in every other period, it is demonstrated that analogous results are obtainable for the case in which amounts of stock ordered at unit prices c_k, c_{k+1}, and c_{k+2} $(c_k > c_{k+1} > c_{k+2})$ are delivered, respectively, k, $k+1$, and $k+2$ periods later.

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I. INTRODUCTION[1]

In this paper we analyze a dynamic inventory problem in which different modes of delivery of ordered stock may be achieved by use of different ordering costs. An example of such a model is the case in which an amount of stock ordered at a unit price c_1 is delivered one period later and another amount ordered at a unit price c_2 $(c_2 < c_1)$ is delivered two periods later.

A basic assumption we make in the present model is that any excess demand is to be backlogged. This is identical to the one made in [1] for the case of constant delivery lag. Besides the ordering costs, there are two more costs operative in the present model: a holding cost depending on the amount of stock at the end of a period and a shortage cost depending on the excess amount of demand over available stock during the period. The holding and shortage cost functions are taken as linear functions for the sake of simplicity. It will be seen later that the results obtained will be readily extended to the case where these two functions are assumed to be convex increasing. If the stock on hand at the beginning of a period is x, then the expected operational costs during the period, exclusive of ordering costs, are given by

$$L(x) = \begin{cases} \displaystyle\int_0^x h(x-t)\, f(t)\, dt + \int_x^\infty p(t-x)\, f(t)\, dt, & x > 0, \\ \displaystyle\int_0^\infty p(t-x)\, f(t)\, dt, & x \le 0, \end{cases} \tag{1}$$

[1] The author gratefully acknowledges his indebtedness to Andrew J. Clark.

where $h(\cdot)$ and $p(\cdot)$ are holding and shortage cost functions, respectively, and the demand quantity during the period is represented by a random variable t with density function $f(t)$. Under the present assumption we have $h(x) = h \cdot x$ and $p(x) = p \cdot x$, where h and p are unit holding and shortage costs, respectively.

In Section II we analyze a model in which an amount of stock ordered at a unit price c_0 is delivered immediately, and another amount ordered at a unit price c_1 $(c_1 < c_0)$ is delivered one period later. In Section III we treat the model of Section II under an assumption that the additional fixed set-up cost is required for any ordering. We also demonstrate that the results of the preceding models are readily extended to the case where amounts of stock ordered at unit prices c_k and c_{k+1} $(c_k > c_{k+1})$, $k \geq 1$, are delivered, respectively, k and $k+1$ periods later. In Section IV we analyze a model in which amounts of stock ordered at unit prices c_k, c_{k+1}, and c_{k+2} $(c_k > c_{k+1} > c_{k+2})$, $k \geq 0$, are delivered, respectively, k, $k+1$, and $k+2$ periods later, under an assumption that the ordering decisions are only made in every other period.

II. TWO MODES OF DELIVERY UNDER LINEAR ORDERING COSTS

In this section we will analyze the optimal ordering policies of a dynamic inventory problem in which an amount of stock ordered at a unit price c_0 is delivered with no delivery lag, and another amount ordered at a unit price c_1 $(c_1 < c_0)$ is delivered one period later.

Starting with the single period model to prepare for the induction, it is clear that we have no ordering at unit price c_1. Next, let us designate by $C_n(x)$ the minimum total expected costs for the n period model when the stock level at the beginning of the initial period is x. If we are going to order amount z_0 at unit price c_0 in the single period model, then we have

$$
\begin{aligned}
C_1(x) &= \min_{z_0 \geq 0} \left\{ c_0 z_0 + L(x + z_0) \right\} \\
&= \min_{x \geq x} \left\{ c_0(w - x) + L(w) \right\}
\end{aligned}
\tag{2}
$$

where $L(x)$ is defined by (1). To determine the optimal ordering level x_1, we obtain from (2) the following expression:

$$
F_1(w) = c_0 w + L(w) \quad . \tag{3}
$$

Since $L'(x)$ is increasing in x, $F_1'(w)$ is increasing in w, and it tends to $c_0 + h > 0$ as w tends to infinity. On the other hand, we have $F_1'(w) = c_0 - p$ for $w \leq 0$. Therefore, if we assume that

$$
c_0 - p < 0 , \tag{4}
$$

then there exists a unique positive x_1 such that $F_1'(x_1) = 0$. It is easily seen that we order up to x_1 if $x < x_1$, and we order nothing if $x \geq x_1$. $C_1(x)$ is accordingly given by

$$C_1(x) = \begin{cases} c_0(x_1 - x) + L(x_1), & x < x_1, \\ L(x), & x \geq x_1. \end{cases} \tag{5}$$

We notice that $C_1(x)$ is convex in x, and $C_1'(x) \geq -c_0$ for all x.

We next proceed to the two period model. If we will decide to order amounts z_0 and z_1 at unit prices c_0 and c_1, respectively, then we have

$$C_2(x) = \min_{\substack{z_0 \geq 0 \\ z_1 \geq 0}} \left\{ c_0 z_0 + c_1 z_1 + L(x + z_0) + \alpha \int_0^\infty C_1(x + z_1 + z_1 - t)\, f(t)\, dt \right\} \tag{6}$$

where α is the discount factor. If we make substitutions $w = x + z_0$, $v = x + z_0 + z_1$, and $c = c_0 - c_1$, then (6) may be written as

$$C_2(x) = \min_{v \geq w \geq x} \left\{ c(w - x) + L(w) + c_1(v - x) + \alpha \int_0^\infty C_1(v - t)\, f(t)\, dt \right\}. \tag{7}$$

Now let us consider the following minimization:

$$\widetilde{L}(x) = \min_{w \geq x} \left\{ c(w - x) + L(w) \right\} . \tag{8}$$

Since the right-hand side of (8) is identical to that of (2) except c_0 replaced by c, we may immediately conclude that, under assumption (4), there exists a unique positive $\hat{x}$ such that $\hat{x} > x_1$ (since $c < c_0$) and

$$c + L'(\hat{x}) = c_0 - c_1 + L'(\hat{x}) = 0 . \tag{9}$$

It follows that $w = \hat{x}$ for $x < \hat{x}$, and $w = x$ for $x \geq \hat{x}$; consequently, $\widetilde{L}(x)$ is given by

$$\begin{aligned} \widetilde{L}(x) &= c(\hat{x} - x) + L(\hat{x}) , && x < \hat{x} , \\ &= L(x) , && x \geq \hat{x} . \end{aligned} \tag{10}$$

Returning to (7) we consider the following cases to accomplish the desired minimization.

<u>Case (a) $x \geq \hat{x}$</u>

In this case we will have $w = x$ as it has been shown by minimization of (8), and (7) may be rewritten as

$$C_2(x) = L(x) + \min_{v \geq x} \left\{ c_1(v - x) + \alpha \int_0^\infty C_1(v - t) f(t)\, dt \right\} . \tag{11}$$

Case (b) $x < \hat{x}$

In this case we will have $w = \hat{x}$ as it has been shown by minimization of (8), if an additional restriction that $v \geq w$ does permit w to assume the value $\hat{x}$; that is to say, if $v \geq \hat{x}$. Otherwise we must have $w = v < \hat{x}$ due to the convexity of $\tilde{L}(x)$ (we may remark here that this does not necessarily follow if $\tilde{L}(x)$ is not convex). Therefore, we have:

1. If $v \geq \hat{x}$,

$$C_2(x) = c(\hat{x} - x) + L(\hat{x}) + \min_{v \geq \hat{x}} \left\{ c_1(v - x) + \alpha \int_0^\infty C_1(v - t) f(t)\, dt \right\} . \tag{12}$$

2. If $v < \hat{x}$,

$$C_2(x) = \min_{\hat{x} > v \geq x} \left\{ c(v - x) + L(v) + c_1(v - x) + \alpha \int_0^\infty C_1(v - t) f(t)\, dt \right\}$$

$$= c(\hat{x} - x) + L(\hat{x}) + \min_{\hat{x} > v \geq x} \left\{ c_1(v - x) + c(v - \hat{x}) + L(v) - L(\hat{v}) + \alpha \int_0^\infty C_1(v - t) f(t)\, dt \right\} . \tag{13}$$

If we define $\Lambda(v)$ by

$$\begin{aligned} \Lambda(v) &= c(v - \hat{x}) + L(v) - L(\hat{x}), & v < \hat{x}\,, \\ &= 0\,, & v \geq \hat{x}\,, \end{aligned} \tag{14}$$

then, by use of $\widetilde{L}(x)$ and $\Lambda(v)$, we may represent expressions (11), (12), and (13) by a single expression

$$C_2(x) = \widetilde{L}(x) + \min_{v \geq x} \left\{ c_1(v - x) + \Lambda(v) + \alpha \int_0^{\infty} C_1(v - t) f(t)\, dt \right\} . \tag{15}$$

It is essential to notice that $\Lambda(v)$ is a continuous convex function of v, since, otherwise, we will find a serious difficulty in achieving the minimization involved in (15).

To determine the optimal ordering level for the two period model, we obtain from (15) the following quantity:

$$F_2(v) = c_1 v + \Lambda(v) + \alpha \int_0^{\infty} C_1(v - t) f(t)\, dt \ . \tag{16}$$

The derivative $F_2'(v)$ tends to $c_1 + \alpha h > 0$ as v tends to infinity and, at $v = \hat{x}$, it is given by

$$F_2'(\hat{x}) = c_1 + \alpha \int_0^{\infty} C_1'(\hat{x} - t) f(t)\, dt \ . \tag{17}$$

Two cases may now be considered, according to the sign of $F_2'(\hat{x})$:

Case (a) $F_2'(\hat{x}) < 0$

Since $F_2(v)$ is convex in v due to convexity of $\Lambda(v)$ and $C_1(v)$, there exists a unique positive x_2 such that $x_2 > \hat{x}$ and

$$F_2'(x_2) = 0 = c_1 + \alpha \int_0^\infty C_1'(x_2 - t) f(t)\, dt \ . \tag{18}$$

It follows that we have in (15) $v = x_2$ if $x < x_2$ and $v = x$ if $x \geq x_2$. Therefore, choosing appropriate forms of $\tilde{L}(x)$ and $\Lambda(v)$ from (10) and (14), respectively, for each x, we obtain from (15)

$$\begin{aligned} C_2(x) &= c_0(\hat{x} - x) + c_1(x_2 - \hat{x}) + L(\hat{x}) + \alpha \int_0^\infty C_1(x_2 - t) f(t)\, dt \ , && x < \hat{x} \ , \\ &= c_1(x_2 - x) + L(x) + \alpha \int_0^\infty C_1(x_2 - t) f(t)\, dt \ , && \hat{x} \leq x < x_2 \ , \\ &= L(x) + \alpha \int_0^\infty C_1(x - t) f(t)\, dt \ , && x \geq x_2 \ . \end{aligned} \tag{19}$$

It follows that the optimal policy is to order:

1. for $x < \hat{x}$, amount $\hat{x} - x$ at c_0 and amount $x_2 - \hat{x}$ at c_1 ;
2. for $\hat{x} \leq x < x_2$, amount $x_2 - x$ at c_1 ; (20)
3. for $x \geq x_2$, none.

The derivative of $C_2(x)$ is given by

$$\begin{aligned} C_2'(x) &= -c_0 \ , && x < \hat{x} \ , \\ &= -c_1 + L'(x), && \hat{x} < x < x_2 \ , \\ &= L'(x) + \alpha \int_0^\infty C_1'(x - t) f(t)\, dt \ , && x > x_2 \ . \end{aligned} \tag{21}$$

It is noticed that $C_2(x)$ is convex in x, $C_2'(x) \geq -c_0$ for all x, and $C_1'(x) \geq C_2'(x)$ for $x \leq x_2$.

Let us next consider an equation

$$F(x) = c_1 + \alpha \int_0^\infty g(x-t) f(t)\, dt = 0 \tag{22}$$

where $g(x)$ is defined by

$$\begin{aligned} g(x) &= -c_0, & x &< \hat{x}, \\ &= -c_1 + L'(x), & x &> \hat{x}. \end{aligned} \tag{23}$$

We remark that $g(x) \leq C_2'(x)$ and $g(x) \leq C_1'(x)$ for all x. As x tends to infinity $F(x)$ tends to $c_1(1-\alpha) + \alpha h > 0$, and $F(\hat{x}) = c_1 - \alpha c_0$. If $c_1 - \alpha c_0 \geq 0$, then we have $F_2'(\hat{x}) \geq c_1 - \alpha c_0 \geq 0$ from (17) due to the fact that $C_1'(x) \geq -c_0$ for all x. This is contradictory to the assumption of this case. Therefore, we must have $F(\hat{x}) = c_1 - \alpha c_0 < 0$, and, since $F(x)$ is increasing in x, there exists a unique positive $\bar{x}$ such that $\bar{x} > \hat{x}$ and

$$F(\bar{x}) = 0 = c_1 + \alpha \int_0^\infty g(\bar{x} - t) f(t)\, dt \quad . \tag{24}$$

It follows that $\bar{x} \geq x_2$ since we have

$$F_2'(\bar{x}) = F_2'(\bar{x}) - F(\bar{x}) = \alpha \int_0^\infty \left\{ C_1'(\bar{x}-t) - g(\bar{x}-t) \right\} f(t)\, dt \geq 0$$

due to the fact that $C_1'(x) \geq g(x)$ for all x.

Case (b) $F_2'(\hat{x}) \geq 0$

We first recall that $F_2'(x)$ is given by

$$F_2'(x) = c_0 + L'(x) + \alpha \int_0^\infty C_1'(x - t) f(t)\, dt$$

for $x < \hat{x}$. At $x = x_1$ we have $F_2'(x_1) = c_0 + L'(x_1) - \alpha c_0 = - \alpha c_0 < 0$. Therefore, there exists a unique x_2 such that $\hat{x} \geq x_2 > x_1$ and

$$F_2'(x_2) = 0 = c_0 + L'(x_2) + \alpha \int_0^\infty C_1'(x_2 - t) f(t)\, dt \ .$$

By use of this x_2 we obtain from (15)

$$C_2(x) = c_0(x_2 - x) + L(x_2) + \alpha \int_0^\infty C_1(x_2 - t) f(t)\, dt \ , \qquad x < x_2 \ , \tag{25}$$

$$= L(x) + \alpha \int_0^\infty C_1(x - t) f(t)\, dt \ , \qquad x \geq x_2 \ .$$

It follows that the optimal policy is to order:

1. for $x < x_2$, amount $x_2 - x$ at c_0,
2. for $x \geq x_2$, none.

(26)

The derivative of $C_2(x)$ is given by

$$
\begin{aligned}
C_2'(x) &= -c_0, & x < x_2, \\
&= L'(x) + \alpha\int_0^\infty C_1'(x-t)f(t)\,dt, & x > x_2.
\end{aligned}
\tag{27}
$$

It is noticed that $C_2(x)$ is convex in x, $C_2'(x) \geq -c_0$ for all x, and $C_1'(x) \geq C_2'(x)$ for $x \leq x_2$. It is also remarked that $C_2'(x) \geq g(x)$ for all x.

Let us next consider an equation

$$F'(x) = c_0(1-\alpha) + L'(x) .$$

As x tends to infinity $F'(x)$ tends to $c_0(1-\alpha) + h > 0$, and, at $x = x_1$, $F'(x_1) = c_0(1-\alpha) + L'(x_1) = -\alpha c_0 < 0$. Therefore, there exists a unique $\hat{x}'$ such that $\hat{x}' > x_1$ and

$$F'(\hat{x}') = 0 = c_0\,(1-\alpha) + L'(\hat{x}') . \tag{28}$$

It follows that $\hat{x}' \geq x_2$, since we have

$$F_2'(\hat{x}') = F_2'(\hat{x}') - F'(\hat{x}') = \alpha\int_0^\infty \left\{C_1'(\hat{x}'-t) + c_0\right\} f(t)\,dt \geq 0$$

due to the fact that $C_1'(x) \geq -c_0$ for all x. At $x = \hat{x}$, $F'(\hat{x}) = c_0 - \alpha c_0 + L'(\hat{x}) = c_1 - \alpha c_0$; hence, it also follows that $\hat{x}' \leq \hat{x}$ if $c_1 - \alpha c_0 \geq 0$, and $\hat{x}' > \hat{x}$ if $c_1 - \alpha c_0 < 0$.

To complete the analysis of the two period model, let us prove that $C_1'(x) \geq C_2'(x)$ for $x \leq \bar{x}$ in both cases. We have already noticed that $C_1'(x) \geq C_2'(x)$ for $x \leq x_2$ in both cases. For $x_2 < x \leq \bar{x}$ we have

$$C_1'(x) - C_2'(x) = -\alpha \int_0^\infty C_1'(x - t) f(t) \, dt \, .$$

Hence, it suffices to show that

$$\alpha \int_0^\infty C_1'(\bar{x} - t) f(t) \, dt \leq 0 \, .$$

To start with, we have, by use of (23) and (24),

$$- c_1 = \alpha \int_0^\infty g(\bar{x} - t) f(t) \, dt = \alpha \int_0^{\bar{x} - \hat{x}} L'(\bar{x} - t) f(t) \, dt - \alpha c_1 \int_0^{\bar{x} - \hat{x}} f(t) \, dt$$

$$- \alpha c_0 \int_{\bar{x} - \hat{x}}^\infty f(t) \, dt \, ,$$

or,

$$\alpha \int_0^{\bar{x} - \hat{x}} L'(\bar{x} - t) f(t) \, dt - \alpha c_0 \int_{\bar{x} - \hat{x}}^\infty f(t) \, dt = - c_1 + \alpha c_1 \int_0^{\bar{x} - \hat{x}} f(t) \, dt \, . \qquad (29)$$

We also have, by use of (5) ,

$$\alpha\int_0^\infty C_1'(\bar{x} - t)f(t)\,dt = \alpha\int_0^{\bar{x} - x_1} L_1'(\bar{x} - t)\,f(t)\,dt - \alpha c_0\int_{\bar{x} - x_1}^\infty f(t)\,dt$$

$$= \alpha\int_0^{\bar{x} - \hat{x}} L'(\bar{x} - t)f(t)\,dt - \alpha c_0\int_{\bar{x} - \hat{x}}^\infty f(t)\,dt + \alpha\int_{\bar{x} - \hat{x}}^{\bar{x} - x_1} L'(\bar{x} - t)f(t)\,dt$$

$$+ \alpha c_0\int_{\bar{x} - \hat{x}}^{\bar{x} - x_1} f(t)\,dt \ ;$$

substituting (29) here, and recalling that $L'(x) \leq -(c_0 - c_1)$ for $x \leq \hat{x}$, we obtain

$$\alpha\int_0^\infty C_1'(\bar{x} - t)f(t)\,dt = -c_1 + \alpha c_1\int_0^{\bar{x} - \hat{x}} f(t)\,dt + \alpha c_0\int_{\bar{x} - \hat{x}}^{\bar{x} - x_1} f(t)\,dt$$

$$+ \alpha\int_{\bar{x} - \hat{x}}^{\bar{x} - x_1} L'(\bar{x} - t)f(t)\,dt$$

$$\leq -c_1 + \alpha c_1\int_0^{\bar{x} - \hat{x}} f(t)\,dt + \alpha c_0\int_{\bar{x} - \hat{x}}^{\bar{x} - x_1} f(t)\,dt - \alpha(c_0 - c_1)\int_{\bar{x} - \hat{x}}^{\bar{x} - x_1} f(t)\,dt$$

$$= -c_1 + \alpha c_1\int_0^{\bar{x} - x_1} f(t)\,dt < 0 \ .$$

This establishes the desired result.

We now will briefly analyze the three period model. To start with, $C_3(x)$ will be defined by an expression identical to (6) except C_1 replaced by C_2. This expression will be reduced to an expression identical to (15), with C_1 replaced by C_2, through the identical arguments used in the two period model based on the cases (a) and (b). Then $F_3(v)$ will be defined by an expression identical to $F_2(v)$ except C_1 replaced by C_2. If we had $F_2'(\hat{x}) < 0$, then we have $F_3'(\hat{x}) < 0$ due to the fact that $C_1'(x) \geq C_2'(x)$ for $x \leq \bar{x}$. We may immediately conclude that there exists a unique x_3 such that $x_3 > \hat{x}$ and $F_3'(x_3) = 0$. Since we have

$$F_3'(x_2) = F_3'(x_2) - F_2'(x_2) = \alpha\int_0^\infty \left\{C_2'(x_2 - t) - C_1'(x_2 - t)\right\} f(t)\,dt \leq 0$$

due to the fact that $C_1'(x) \geq C_2'(x)$ for $x \leq \bar{x}$, and

$$F_3'(\bar{x}) = F_3'(\bar{x}) - F(\bar{x}) = \alpha\int_0^\infty \left\{ C_2'(\bar{x} - t) - g(\bar{x} - t)\right\} f(t)\,dt \geq 0$$

due to the fact that $C_2'(x) \geq g(x)$ for all x, we have $\bar{x} \geq x_3 \geq x_2 \geq \hat{x}$. x_3 and $\hat{x}$ uniquely define the optimal policy which is identical to (20) except x_2 replaced by x_3, and $C_3(x)$ and $C_3'(x)$ are given by expressions identical to (19) and (21), respectively, with x_2 and C_2 replaced by x_3 and C_3, respectively. It will be noticed that $C_3(x)$ is convex in x, $C_3'(x) \geq g(x) \geq -c_0$ for all x, and $C_2'(x) \geq C_3'(x)$ for $x \leq x_3$.

If we had $F_2'(\hat{x}) \geq 0$ with $c_1 - \alpha c_0 < 0$, then we may have $F_3'(\hat{x}) < 0$, since $C_2'(x) \leq C_1'(x)$ for $x \leq \bar{x}$, or we may still have $F_3'(\hat{x}) \geq 0$. The first case has already been analyzed. If we have $c_1 - \alpha c_0 \geq 0$, we must have $F_3'(\hat{x}) \geq 0$. In the second case we have $F_3'(x_1) \leq F_2'(x_1) = -\alpha c_0 < 0$; hence, there exists a unique x_3 such that $\hat{x} \geq x_3 > x_1$ and $F_3'(x_3) = 0$. Since we have

$$F_3'(x_2) = F_3'(x_2) - F_2'(x_2) = \alpha \int_0^\infty \left\{ C_2'(x_2 - t) - C_1'(x_2 - t) \right\} f(t)dt \leq 0$$

and

$$F_3'(\hat{x}') = F_3'(\hat{x}') - F'(\hat{x}') = \alpha \int_0^\infty \left\{ C_2'(\hat{x}' - t) + c_0 \right\} f(t)\, dt \geq 0 ,$$

we have $\hat{x}' \geq x_3 \geq x_2 > x_1$ (we may remark that the upper bound $\hat{x}'$ is immaterial unless we have $c_1 - \alpha c_0 \geq 0$, since, otherwise, we have $\hat{x}' > \hat{x}$). x_3 uniquely determines the optimal policy which is identical to (26) except x_2 replaced by x_3. $C_3(x)$ and $C_3'(x)$ are given by expressions identical to (25) and (27), respectively, except x_2 and C_2 replaced by x_3 and C_3, respectively. It will be noticed that $C_3(x)$ is convex in x, $C_3'(x) \geq g(x) \geq -c_0$ for all x, and $C_2'(x) \geq C_3'(x)$ for $x \leq x_3$. It will finally be shown that we have in both cases

$$C_3'(x) - C_2'(x) = \alpha \int_0^\infty \left\{ C_2'(x - t) - C_1'(x - t) \right\} f(t)dt \leq 0$$

for $x_3 < x \leq \bar{x}$, due to the fact that $C_1'(x) \geq C_2'(x)$ for $x \leq \bar{x}$. Therefore, it will follow that we have $C_2'(x) \geq C_3'(x)$ for $x \leq \bar{x}$.

We now have all the necessary ingredients of an inductive proof, and we summarize the preceding results as

Theorem 1: For each $n \geq 2$ there exists a unique positive x_n which, together with $\hat{x}$ determined by (9), uniquely determines the optimal policy for the n period model as follows:

Case A If $x_n > \hat{x}$, it is optimal to order:

1. for $x < \hat{x}$, amount $\hat{x} - x$ at c_0 and amount $x_n - \hat{x}$ at c_1;
2. for $\hat{x} \leq x < x_n$, amount $x_n - x$ at c_1;
3. for $x \geq x_n$, none.

Case B If $x_n \leq \hat{x}$, it is optimal to order:

1. for $x < x_n$, amount $x_n - x$ at c_0;
2. for $x \geq x_n$, none.

Furthermore, the following properties hold:

(i) x_n is a unique root of an equation

$$c_1 + \Lambda'(v) + \alpha\int_0^\infty C_{n-1}'(v - t)f(t)dt = 0$$

where $\Lambda(v)$ is defined by (14).

(ii) Case A $x_n > \hat{x}$:

$$\begin{aligned} C_n'(x) &= -c_0, & x &< \hat{x}, \\ &= -c_1 + L'(x) & \hat{x} &< x < x_n, \\ &= L'(x) + \alpha\int_0^\infty C_{n-1}'(x - t)f(t)\,dt, & x &> x_n. \end{aligned}$$

<u>Case B</u> $\quad x_n \le \hat{x}$:

$$C_n'(x) = -c_0 , \qquad x < x_n ,$$

$$= L'(x) + \alpha \int_0^{\infty} C_{n-1}'(x-t) f(t)\, dt , \qquad x > x_n .$$

(iii) $C_n(x)$ is a convex function of x and $C_n'(x) \ge g(x) \ge -c_0$ for all x, where $g(x)$ is defined by (23). The second derivative of $C_n(x)$ exists everywhere except possibly for $x = \hat{x}$ and x_n where the right-and left-hand second derivatives exist.

(iv) <u>Case A</u> $\quad c_1 - \alpha c_0 < 0$:

$$x_{n-1} \le x_n \le \bar{x}$$

where $\bar{x}$ is determined by (24); there also exists a unique $i \ge 2$ such that $x_{i-1} \le \hat{x} < x_i$.

<u>Case B</u> $\quad c_1 - \alpha c_0 \ge 0$:

$$x_{n-1} \le x_n \le \hat{x}' \le \hat{x}$$

where $\hat{x}'$ is determined by (28).

(v) $\quad C_{n-1}'(x) \ge C_n'(x)$ for $x \le \bar{x}$.

We remark that these results are similar to results obtained in [1] for the dynamic inventory model with constant delivery lag. It may also be remarked that the above results may be readily extended to the case in which $h(\cdot)$ and $p(\cdot)$ are assumed to be convex increasing, and the right-hand side of (1) may be differentiated twice under the integral signs.

III. INCLUSION OF FIXED ORDER COSTS

In this section we first consider the identical inventory model of the preceding section with the exception of a fixed set-up cost K which we assume to be charged whenever any amount of ordering is made. Therefore, if we define

$$\begin{aligned} K(x) &= K \quad , \qquad x > 0 \, , \\ &= 0 \quad , \qquad x \leq 0 \, , \end{aligned} \tag{30}$$

then the total ordering cost in the present model is given by $c_0 z_0 + c_1 z_1 + K(z_0 + z_1)$ when amounts z_0 and z_1 are ordered at unit prices z_0 and z_1, respectively.

An essential concept in the subsequent analysis of this section is that of K - convexity which has been defined and utilized by Scarf [2] in proving the optimality of (S, s) policies for the dynamic inventory model with constant delivery lag (inclusive of no delivery lag). Specifically, for a given $K \geq 0$, a function $C(x)$ is said to be K - convex if

$$K + C(x + a) - C(x) - \frac{a}{b} \left\{ C(x) - C(x - b) \right\} \geq 0 \tag{31}$$

for all x, all positive a, and all positive b smaller than a positive constant M. It is readily verified that K - convexity has the following properties [2, p. 199]:

(i) 0 - convexity is equivalent to the ordinary convexity.

(ii) If $C(x)$ is K - convex, then $C(x + h)$ is K - convex for all h.

(iii) If $C(x)$ and $D(x)$ are K - convex and L - convex, respectively, then $AC + BD$ is $(AK + BL)$ - convex when A and B are positive. This property may be extended to denumerable sums and integrals.

Returning to our model, we have no ordering at unit price c_1 in the single period model. Therefore, we have

$$C_1(x) = \min_{z_0 \geq 0} \left\{ c_0 z_0 + K(z_0) + L(x + z_0) \right\}$$

Since this expression is identical to the one in the case of no delivery lag which has been analyzed by Scarf in [2], we immediately know that the optimal policy is of (S, s) type and $C_1(x)$ is K - convex.

Let us now proceed to the n period model, assuming that $C_{n-1}(x)$ is K - convex. If we are going to order amounts z_0 and z_1 at unit prices c_0 and c_1, respectively, in this period, then we have

$$C_n(x) = \min_{\substack{z_0 \geq 0 \\ z_1 \geq 0}} \left\{ c_0 z_0 + c_1 z_1 + K(z_0 + z_1) + L(x + z_0) + \alpha \int_0^\infty C_{n-1}(x + z_0 + z_1 - t) f(t) dt \right\} \tag{32}$$

Making substitutions $w = x + z_0$, $v = x + z_0 + z_1$, and $c = c_0 - c_1$, as we did in the previous section, we can rewrite (32) as

$$C_n(x) = \min_{v \geq w \geq x} \left\{ c(w - x) + L(w) + c_1(v - x) + K(v - x) + \alpha \int_0^\infty C_{n-1}(v - t) f(t) dt \right\}. \tag{33}$$

We now see that (33) may be reduced to

$$C_n(x) = \widetilde{L}(x) + g_n(x) \tag{34}$$

where $g_n(x)$ is given by

$$g_n(x) = \min_{v \geq x} \left\{ c_1(v - x) + K(v - x) + \Lambda(v) + \alpha \int_0^\infty C_{n-1}(v - t) f(t)\, dt \right\} , \tag{35}$$

through the identical arguments applied to reduce (7) to (15) in the two period model of the previous section. We recall that $\widetilde{L}(x)$ and $\Lambda(v)$ are defined by (10) and (14), respectively, and both are convex functions. The quantity essential in achieving minimization in (35) is given by

$$G_n(v) = c_1 v + \Lambda(v) + \alpha \int_0^\infty C_{n-1}(v - t)\, f(t)\, dt \quad . \tag{36}$$

Since $C_{n-1}(v)$ is K - convex, and $c_1 v$ and $\Lambda(v)$ are convex, it is easy to see from properties (i), (ii), and (iii) above that $G_n(v)$ is K - convex. It then follows that

$$K + G_n(v + a) - G_n(v) - a\, G_n'(v) \geq 0$$

for all v and all $a \geq 0$ which, in turn, implies that there exists a unique pair (S_n , s_n) such that $S_n > s_n$, $G_n(S_n)$ is the minimum value of $G_n(v)$, and $K + G_n(S_n) = G_n(s_n)$. Based on this result we obtain from (35)

$$\begin{aligned} g_n(x) &= K - c_1 x + G_n(S_n) , && x < s_n , \\ &= - c_1 x + G_n(x) , && x \geq s_n . \end{aligned} \tag{37}$$

The explicit form of $C_n(x)$ now may be determined by use of (10), (14), (34), (36), and (37).

Case (a) $\hat{x} < s_n$

$$C_n(x) = K + c_0(\hat{x} - x) + c_1(S_n - \hat{x}) + L(\hat{x}) + \alpha\int_0^\infty C_{n-1}(S_n - t)f(t)\,dt\,, \qquad x < \hat{x}\,,$$

$$= K + c_1(S_n - x) + L(x) + \alpha\int_0^\infty C_{n-1}(S_n - t)f(t)\,dt\,, \qquad \hat{x} \le x < s_n\,,$$

$$= L(x) + \alpha\int_0^\infty C_{n-1}(x - t)f(t)\,dt\,, \qquad x \ge s_n\,.$$

The optimal policy in this case is to order:

1. for $x < \hat{x}$, amount $\hat{x} - x$ at c_0, and amount $S_n - \hat{x}$ at c_1;
2. for $\hat{x} \le x < s_n$, amount $S_n - x$ at c_1; (38)
3. for $x \ge s_n$, none.

Case (b) $s_n \le \hat{x} < S_n$

$$C_n(x) = K + c_0(\hat{x} - x) + c_1(S_n - \hat{x}) + L(\hat{x}) + \alpha\int_0^\infty C_{n-1}(S_n - t)f(t)\,dt\,, \qquad x < s_n\,,$$

$$= L(x) + \alpha\int_0^\infty C_{n-1}(x - t)f(t)\,dt\,, \qquad x \ge s_n\,.$$

The optimal policy in this case is to order:

1. for $x < s_n$, amount $\hat{x} - x$ at c_0, and amount $S_n - \hat{x}$ at c_1;
2. for $x \ge s_n$, none. (39)

<u>Case (c) $\hat{x} \geq S_n$</u>

$$C_n(x) = K + c_0 (S_n - x) + L(S_n) + \alpha \int_0^{\infty} C_{n-1}(S_n - t) f(t)\, dt \,, \qquad x < s_n,$$

$$= L(x) + \alpha \int_0^{\infty} C_{n-1}(x - t) f(t)\, dt \,, \qquad x \geq s_n \,.$$

The optimal policy in this case is to order:

$$\begin{aligned} &1. \quad \text{for } x < s_n \text{, amount } S_n - x \text{ at } c_0 \,, \\ &2. \quad \text{for } x \geq s_n \text{, none.} \end{aligned} \tag{40}$$

The final step in the induction is to establish the K - convexity of $C_n(x)$. Since $\widetilde{L}(x)$ is convex, and $C_n(x)$ is given by (34), it suffices to establish the K - convexity of $g_n(x)$ which is expressed in terms of a K - convex function $G_n(x)$ by (37). But this is exactly what has been done by Scarf [2, p. 200]. Therefore, this completes the analysis of the optimal policies in the present model.

Let us next consider the case in which amounts of stock ordered at unit prices c_k and c_{k+1}, $k \geq 1$ $(c_k > c_{k+1})$, are delivered, respectively, k and $k+1$ periods later. If amounts z_k and z_{k+1} are ordered at unit prices c_k and c_{k+1}, respectively, in the n period model, then we have for $n > k+1$

$$C_n(x, x_1, x_2, \cdots, x_k) = \min_{\substack{z_k \geq 0 \\ z_{k+1} \geq 0}} \Big\{ c_k z_k + c_{k+1} z_{k+1} + K(z_k + z_{k+1}) + L(x) + \alpha \int_0^{\infty} C_{n-1}(x + x_1 - t, x_2, \cdots, x_k + z_k, z_{k+1}) f(t)\, dt \Big\} \,, \tag{41}$$

where $C_n(x, x_1, x_2 \cdots, x_k)$ represents the expected total minimum cost for the n period model when the stock level at the beginning of the initial period is x, and x_i is the amount of outstanding order to be delivered i periods later. It is obvious that there is no ordering in the k period model. Therefore, if we define $L_0(x) = L(x)$, and

$$L_i(x) = \alpha \int_0^\infty L_{i-1}(x-t) f(t)\, dt , \qquad i \geq 1 , \tag{42}$$

then we have

$$C_k(x, x_1, x_2, \cdots, x_{k-1}) = L(x) + L_1(x + x_1) + \cdots + L_{k-1}(x + x_1 + \cdots + x_{k-1}) . \tag{43}$$

We now assume that we have for $n > k+1$

$$C_n(x, x_1, \cdots, x_k) = L(x) + L_1(x) + \cdots + L_{k-1}(x + x_1 + \cdots + x_{k-1}) + H_n(u) \tag{44}$$

where $u = x + x_1 + \cdots + x_k$, and

$$H_n(u) = \min_{\substack{z_k \geq 0 \\ z_{k+1} \geq 0}} \left\{ c_k z_k + c_{k+1} z_{k+1} + K(z_k + z_{k+1}) + L_k(u + z_k) + \alpha \int_0^\infty H_{n-1}(u + z_k + z_{k+1} - t) f(t)dt \right\} . \tag{45}$$

As a special case, we have for $n = k+1$

$$H_{k+1}(u) = \min_{z_k \geq 0} \left\{ c_k z_k + K(z_k) + L_k(u + z_k) \right\} .$$

The validity of (44) and (45) may be readily verified by a direct substitution. If we make substitutions $u + z_k = w$, $u + z_k + z_{k+1} = v$, and $c = c_k - c_{k+1}$, then (45) may be rewritten as

$$H_n(u) = \min_{v \geq w \geq u} \left\{ c(w - u) + L_k(w) + c_{k+1}(v - x) + K(v - u) \right. \tag{46}$$
$$\left. + \alpha \int_0^{\infty} H_{n-1}(v - t) f(t)\, dt \right\} .$$

We notice that (46) is essentially identical to (33) except $L(w)$ replaced by $L_k(w)$ which is again convex due to the convexity of $L(w)$. Therefore, all the arguments used in the analysis of the preceding model are applicable here, and we may immediately conclude that the optimal policies in the present model are essentially identical to those of the preceding model (see (38), (39), and (40)).

Furthermore, if we assume that $K(x) \equiv 0$ for all x in (41), then (41) represents the expected total cost for the n period model in the case of linear ordering costs. We notice that (46) which was obtained from (41) is essentially identical to (7) under the assumption $K(x) \equiv 0$ for all x. Therefore, it may be concluded that the results of Theorem 1 are readily extended to the present model with no fixed cost.

IV. THREE MODES OF DELIVERY

In this section we consider the case in which amounts of stock ordered at unit prices c_k, c_{k+1} and c_{k+2} $(c_k > c_{k+1} > c_{k+2})$ are delivered, respectively, k, $k+1$, and $k+2$ periods later. In order to apply our technique employed in the preceding sections we assume that ordering of stock is only considered in every other period. Under this assumption, if we order amounts z_0, z_1, and z_2 at units prices c_0, c_1, and c_2, respectively, in the n period model, then we have for $n \geq 3$

$$C_n(x) = \min_{\substack{z_0 \geq 0 \\ z_1 \geq 0 \\ z_2 \geq 0}} \Big\{ K(z_0 + z_1 + z_2) + c_0 z_0 + c_1 z_1 + c_2 z_2 + L(x + z_0) + L_1(x + z_0 + z_1) + C_{n-2,2}(x + z_0 + z_1 + z_2) \Big\} \tag{47}$$

where L_1 is defined by (42), and $C_{n-2,j}$ is similarly defined by $C_{n-2,0}(x) = C_{n-2}(x)$ and

$$C_{n-2,j}(x) = \alpha \int_0^{\infty} C_{n-2,j-1}(x - t) f(t)\, dt \ . \tag{48}$$

For $n = 1$ and 2, we have, respectively,

$$C_1(x) = \min_{z_0 \geq 0} \Big\{ K(z_0) + c_0 z_0 + L(x + z_0) \Big\} \tag{49}$$

and

$$C_2(x) = \min_{\substack{z_0 \geq 0 \\ z_1 \geq 0}} \left\{ K(z_0 + z_1) + c_0 z_0 + c_1 z_1 + L(x + z_0) + L_1 (x + z_0 + z_1) \right\} . \quad (50)$$

If we make substitutions $x + z_0 = w$, $x + z_0 + z_1 = v$, $x + z_0 + z_1 + z_2 = u$, $\bar{c}_0 = c_0 - c_1$, and $\bar{c}_1 = c_1 - c_2$ in (47), then (47) may be rewritten as

$$C_n(x) = \min_{u \geq v \geq w \geq x} \left\{ \bar{c}_0(w - x) + L(w) + \bar{c}_1(v - x) + L_1(v) + c_2(u - x) \right. \quad (51)$$
$$\left. + K(u - x) + C_{n-2,\, 2}(u) \right\} .$$

We first consider the following minimization:

$$\widetilde{L}(x) = \min_{w \geq x} \left\{ \bar{c}_0(w - x) + L(w) \right\} . \quad (52)$$

Since (52) is identical to (8), there exists a unique positive $\widetilde{x}$ such that $\bar{c}_0 + L'(\widetilde{x}) = 0$, and $\widetilde{x}$ uniquely determines $\widetilde{L}(x)$. If we recall, at this point, the technique which was applied to reduce (7) to (15) in Section II, then it will be readily seen that (51) may be reduced by the same technique to (53)

$$C_n(x) = \widetilde{L}(x) + \min_{u \geq v \geq x} \left\{ \bar{c}_1(v - x) + \Lambda(v) + L_1(v) + c_2(u - x) \right. \quad (53)$$
$$\left. + K(u - x) + C_{n-2,\, 2}(u) \right\}$$

where $\Lambda(v)$ is given by

$$\Lambda(v) = \bar{c}_0 (v - \tilde{x}) + L(v) - L(\tilde{x}) , \qquad v < \tilde{x} , \tag{54}$$
$$= 0 , \qquad v \geq \tilde{x} .$$

We notice that $\Lambda(v)$ is convex in v .

Let us next consider the following minimization:

$$\tilde{M}(x) = \min_{v \geq x} \left\{ \bar{c}_1 (v - x) + \Lambda(v) + L_1(v) \right\} . \tag{55}$$

It is easily seen that there exists a unique positive root $\hat{x}$ for the equation

$$M'(v) = \bar{c}_1 + \Lambda'(v) + L_1'(v) = 0 \tag{56}$$

since $M'(v)$ is strictly increasing in v, $M'(0+) = \bar{c}_1 + \bar{c}_0 + L'(0) + L_1'(0)$ $= c_0 - c_2 - p - \alpha p < 0$, and $M'(v)$ tends to $\bar{c}_1 + \alpha h > 0$ as v tends to infinity. It follows that $\hat{x}$ uniquely determines $\tilde{M}(x)$ as

$$\tilde{M}(x) = \bar{c}_1 (x - \hat{x}) + \Lambda(\hat{x}) + L_1(\hat{x}) , \qquad x < \hat{x} , \tag{57}$$
$$= \Lambda(x) + L_1(x), \qquad x \geq \hat{x} .$$

$\tilde{M}(x)$ is clearly convex in x. If we again apply our technique to the minimization in the right-hand side of (53), then (53) may be further reduced to (58)

$$C_n(x) = \tilde{L}(x) + \tilde{M}(x) + \min_{u \geq x} \Big\{ c_2(u - x) + K(u - x) + \tilde{\Lambda}(u) + C_{n-2,2}(u) \Big\} \tag{58}$$

where $\tilde{\Lambda}(u)$ is given by

$$\begin{aligned} \tilde{\Lambda}(u) &= \bar{c}_1(u - \hat{x}) + \Lambda(u) - \Lambda(\hat{x}) + L_1(u) - L_1(\hat{x}), & u < \hat{x}, \\ &= 0, & u \geq \hat{x}. \end{aligned} \tag{59}$$

We again notice that $\tilde{\Lambda}(u)$ is convex in u. It is clear from (58) that the quantity essential in determination of the optimal policy for the n period model is given by

$$G_n(u) = c_2 u + \tilde{\Lambda}(u) + C_{n-2,2}(u) . \tag{60}$$

Since (60) is analogous to (36) in Section III, we may anticipate analogous results in the present model. In fact, if we assume that $K(x) \equiv 0$ for all x and $\hat{x} > \tilde{x}$, then the following theorem which is analogous to Theorem 1 is seen to be true:

<u>Theorem 2</u>: For each $n \geq 3$ there exists a unique positive x_n which, together with $\tilde{x}$ and $\hat{x}$, uniquely determines the optimal policy for the n period model as follows:

<u>Case A</u> If $x_n > \hat{x}$, it is optimal to order:

1. for $x < \tilde{x}$, amount $\tilde{x} - x$ at c_0, amount $\hat{x} - \tilde{x}$ at c_1, and amount $x_n - \hat{x}$ at c_2;
2. for $\tilde{x} \leq x < \hat{x}$, amount $\hat{x} - x$ at c_1, and amount $x_n - \hat{x}$ at c_2;

3. for $\hat{x} \le x < x_n$, amount $x_n - x$ at c_2 ;
4. for $x \ge x_n$, none.

Case B If $\hat{x} \ge x_n > \tilde{x}$, it is optimal to order:

1. for $x < \tilde{x}$, amount $\tilde{x} - x$ at c_0 , and amount $x_n - \tilde{x}$ at c_1 ;
2. for $\tilde{x} \le x < x_n$, amount $x_n - x$ at c_1 ;
3. for $x \ge x_n$, none.

Case C If $x_n \le \tilde{x}$, it is optimal to order:

1. for $x < x_n$, amount $x_n - x$ at c_0 ;
2. for $x \ge x_n$, none.

Furthermore, the following properties hold:

(i) x_n is a unique root of an equation $F_n'(u) = 0$ where $F_n'(u)$ is given by

$$
\begin{aligned}
F_n'(u) &= c_0 + L'(u) + L_1'(u) + C_{n-2,2}'(u) , && u < \tilde{x} , \\
&= c_1 + L_1'(u) + C_{n-2,2}'(u) , && \tilde{x} < u < \hat{x} , \\
&= c_2 + C_{n-2,2}'(u) , && u > \hat{x} ,
\end{aligned}
\tag{61}
$$

(ii) Case A $x_n > \hat{x}$:

$$
\begin{aligned}
C_n'(x) &= - c_0 , && x < \tilde{x} , \\
&= - c_1 + L'(x), && \tilde{x} < x < \hat{x} , \\
&= - c_2 + L'(x) + L_1'(x) , && \hat{x} < x < x_n, \\
&= L'(x) + L_1'(x) + C_{n-2,2}'(x) , && x > x_n .
\end{aligned}
$$

Case B $\hat{x} \ge x_n > \tilde{x}$:

$$
\begin{aligned}
C_n'(x) &= - c_0 , && x < \tilde{x} , \\
&= - c_1 + L'(x), && x < x < x_n, \\
&= L'(x) + L_1'(x) + C_{n-2,2}'(x) , && x > x_n .
\end{aligned}
$$

Case C $\quad x_n \le \tilde{x}$:

$$C_n'(x) = -c_0\,, \qquad x < x_n\,,$$
$$= L'(x) + L_1'(x) + C_{n-2,2}'(x)\,, \qquad x > x_n\,.$$

(iii) $C_n(x)$ is a convex function of x, and $C_n'(x) \ge g(x) \ge \hat{g}(x) \ge -c_0$ for all x, where $\hat{g}(x)$ and $g(x)$ are given as follows:

$$\hat{g}(x) = -c_0\,, \qquad x < \tilde{x}\,,$$
$$= -c_1 + L'(x)\,, \qquad x > \tilde{x}\,;$$
$$g(x) = -c_0\,, \qquad x < \tilde{x}\,,$$
$$= -c_1 + L'(x)\,, \qquad \tilde{x} < x < \hat{x}\,,$$
$$= -c_2 + L'(x) + L_1'(x)\,, \qquad x > \hat{x}\,.$$

The second derivative of $C_n(x)$ exists everywhere except possibly for $x = \tilde{x}, \hat{x}$, and x_n where the right- and left-hand second derivatives exist.

(iv) Let us designate by $\tilde{x}'$, $\hat{x}'$, and $\bar{x}$ the unique roots of the equations

$$\tilde{F}(x) = c_0(1 - \alpha^2) + L'(x) + L_1'(x) = 0\,, \tag{62}$$

$$F'(x) = c_1 + L_1'(x) + \hat{g}_2(x) = 0\,, \tag{63}$$

and

$$F(x) = c_2 + g_2(x) = 0\,, \tag{64}$$

respectively, where $\hat{g}_2(x)$ and $g_2(x)$ are defined by $\hat{g}(x)$ and $g(x)$, respectively, in the identical way as $L_2(x)$ is defined by $L(x)$ in (42).

Case A $\quad c_2 + \hat{g}_2(\hat{x}) < 0$.

In this case we have

$$x_{n-2} \le x_n \le \bar{x}$$

where $\bar{x}$ is determined by (64); there also exist integers i and j such that $i \ge j$, $i \ge 4$, $j \ge 3$, $x_{i-2} \le \hat{x} < x_i$, and $x_{j-2} \le \tilde{x} < x_j$.

<u>Case B</u> $\quad c_1 + L_1'(\tilde{x}) + \hat{g}_2(\tilde{x}) < 0 \leq c_2 + \hat{g}_2(\hat{x})$.

In this case we have

$$x_{n-2} \leq x_n \leq \hat{x}' \leq \hat{x}$$

where $\hat{x}'$ is determined by (63); there also exists an integer k such that $k \geq 3$ and $x_{k-2} \leq \tilde{x} < x_k$.

<u>Case C</u> $\quad 0 \leq c_1 + L_1'(\tilde{x}) + \hat{g}_2(\tilde{x})$.

In this case we have

$$x_{n-2} \leq x_n \leq \tilde{x}' \leq \tilde{x}$$

where $\tilde{x}'$ is determined by (62).

(v) $\quad C_n'(x) \leq C_{n-2}'(x)$ for $x \leq \tilde{x}$.

The results for $n = 1$ and 2 will be readily obtained by use of (49) and (50). The proof of the above theorem will be similarly constructed as for Theorem 1, and, therefore, we will omit it. We also remark that, if we assume $\hat{x} \leq \tilde{x}$, similar results as above will also be obtained.

If $K(x)$ is given by (30), then the essential part of analysis of the optimal policy is to establish the K - convexity of $G_n(u)$ given by (60). This may be done inductively as in Section III, and the optimal policies can be determined in forms similar to those in Section III (see (38), (39), and (40)). In the present model there will be more possibilities in the form of the optimal policy than in Section III, but we will present no further details. Finally we remark that these results may be readily extended to the case in which we have $k \geq 1$ as we have seen it in Section III.

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